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CANONICAL MODULI AND GENERAL SOLUTION OF EQUATIONS OF A TWO-DIMENSIONAL STATIC PROBLEM OF ANISOTROPIC ELASTICITY

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Equations of a two-dimensional static problem of anisotropic elasticity are brought to a simple form with the use of orthogonal and affine transformations of coordinates and corresponding transformations of mechanical quantities. It is proved that an arbitrary matrix of elasticity moduli containing six independent components can be always converted by a congruent transformation to a matrix with two independent components, which are called the canonical moduli. Depending on the relations between the canonical moduli, the determinant of the matrix of operators of equations in displacements is presented as a product of various quadratic terms. A general presentation of the solution of equations in displacements in the form of a linear combination of the first derivatives of two quasi-harmonic functions satisfying two independent equations is given. A symmetry operator (*i.e., a formula of production of new solutions*) *is found to correspond to each presentation. In a three-dimensional case, the matrix of elasticity moduli with 21 independent components is congruent to a matrix with 12 independent canonical moduli.*

Key words: *orthogonal and affine transformations, anisotropy, elasticity moduli, canonical moduli, general solution, symmetry operators, diagonalization of an elliptical system.*

It was shown [1] that, in the case with arbitrary anisotropy, the equations in displacements

$$A_{i(kl)j}\,\partial_{kl}u_j + F_i = 0, \qquad i,j,k,l = 1,2,3, \tag{1}$$

where $A_{i(kl)j} = (A_{iklj} + A_{ilkj})/2$, $A_{iklj} = A_{kilj} = A_{ljik}$ is the tensor of the fourth rank of the elasticity moduli, u_j is the displacement vector, F_i is the vector of the volume forces, and ∂_k is the derivative with respect to the coordinate x_k (summation is performed over the repeated letter subscripts) under the general affine transformation of coordinates

$$x_i = \alpha_i + \alpha_{ij}y_j, \qquad |\alpha_{ij}| \neq 0, \qquad y_k = \beta_k + \beta_{ki}x_i; \tag{2}$$

$$\beta_k = -\beta_{ki}\alpha_i, \qquad \beta_{ki}\alpha_{ij} = \delta_{kj}$$

(δ_{kj} is a unit matrix and α_i and α_{ij} are arbitrary real constants) and appropriate transformations of mechanical quantities do not change their form:

$$\tilde{A}_{r(pq)s}\,\tilde{\partial}_{pq}\tilde{u}_s + \tilde{F}_r = 0$$

(the tilde indicates the transformed quantities). The equilibrium equations, generalized Hooke's law, specific strain energy, expressions of strains in terms of displacements, and boundary conditions in stresses and displacements also retain their form in the new variables [1]. With an appropriate choice of the parameters α_{ij} and β_{ki} in the affine transformations (2), it is possible to simplify Eqs. (1) and reduce the number of elasticity moduli in Hooke's law. Some affine transformations of static equations of the linear elasticity theory for anisotropic solids were considered, e.g., in [2–7] (see also the references in [1]).

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The tensor of the elasticity moduli and the coefficients in Eqs. (1) are transformed by the formulas [1]

$$\tilde{A}_{pqrs} = \beta_{pi}\beta_{qj}A_{ijkl}\beta_{rk}\beta_{sl}, \qquad A_{ijkl} = \alpha_{ip}\alpha_{jq}\tilde{A}_{pqrs}\alpha_{kr}\alpha_{ls}; \tag{3}$$

$$\tilde{A}_{r(pq)s} = \beta_{ri}\beta_{sj}A_{i(kl)j}\beta_{pk}\beta_{ql}, \qquad A_{i(kl)j} = \alpha_{ir}\alpha_{js}\tilde{A}_{r(pq)s}\alpha_{kp}\alpha_{lq}.$$

Using the formulas for the transition from two subscripts to one for symmetric tensors in terms of two subscripts,

$$h_{11} = h_1, \qquad h_{22} = h_2, \qquad h_{33} = h_3,$$

$$\sqrt{2}\,h_{23} = \sqrt{2}\,h_{32} = h_4, \qquad \sqrt{2}\,h_{13} = \sqrt{2}\,h_{31} = h_5, \qquad \sqrt{2}\,h_{12} = \sqrt{2}\,h_{21} = h_6,$$

we write Eqs. (3) in the matrix form:

$$A = \hat{\alpha}\tilde{A}\hat{\alpha}', \qquad \tilde{A} = \hat{\beta}A\hat{\beta}', \qquad A^{-1} = \hat{\beta}'\tilde{A}^{-1}\hat{\beta}, \qquad \tilde{A}^{-1} = \hat{\alpha}'A^{-1}\hat{\alpha},$$
$$\tilde{A}^{*} = \hat{\beta}A^{*}\hat{\beta}', \qquad A^{*} = \hat{\alpha}\tilde{A}^{*}\hat{\alpha}' \tag{4}$$

(the prime indicates the operation of matrix transposition). In Eqs. (4), all matrices have a size of 6×6. The matrix A^{-1} of the compliance coefficients is inverse to the matrix A of the elasticity moduli. The matrices A and A^{-1} are symmetric and positively determined. The matrices A^{*} and $\tilde{A}^{*}$ correspond to the coefficients $A_{i(kl)j}$ in Eqs. (1). The matrices corresponding to the tensors

$$\alpha_{ijpq} = (\alpha_{ip}\alpha_{jq} + \alpha_{iq}\alpha_{jp})/2, \qquad \beta_{pqij} = (\beta_{pi}\beta_{qj} + \beta_{pj}\beta_{qi})/2$$

are denoted by $\hat{\alpha}$ and $\hat{\beta}$. The components of the matrices $\hat{\alpha}$ and $\hat{\beta}$ can be found in [1, 6]. In the two-dimensional case, the matrix $\hat{\beta}$ has the form

$$\hat{\beta}_{pi} = \begin{bmatrix} \beta_{11}^2 & \beta_{12}^2 & \sqrt{2}\,\beta_{11}\beta_{12} \\ \beta_{21}^2 & \beta_{22}^2 & \sqrt{2}\,\beta_{21}\beta_{22} \\ \sqrt{2}\,\beta_{11}\beta_{21} & \sqrt{2}\,\beta_{12}\beta_{22} & \beta_{11}\beta_{22} + \beta_{12}\beta_{21} \end{bmatrix}.$$

Equations (4) are congruent transformations of the matrices A, $\tilde{A}$, A^{-1}, $\tilde{A}^{-1}$, A^{*}, and $\tilde{A}^{*}$, which determine the properties of elasticity and Eqs. (1) for arbitrary anisotropic materials. The affine transformations (2) with a non-degenerate matrix $\alpha = [\alpha_{ij}]$ form a group, as well as the congruent transformations (4) [1, 6].

Transformation (2) with the matrix $\beta = [\beta_{ki}]$ can be written as a product of three transformations [1, 8]

$$\beta = \beta^{(3)}\beta^{(2)}\beta^{(1)}, \tag{5}$$

where $\beta^{(1)}$ and $\beta^{(3)}$ are the rotation matrices; $\beta^{(2)}$ is the diagonal matrix: $\beta^{(2)} = \operatorname{diag}(\beta_1,\ \beta_2,\ \beta_3)$, $\beta_i > 0$. With allowance for Eq. (5), we obtain the following presentation for $\hat{\beta}$:

$$\hat{\beta} = \hat{\beta}^{(3)}\hat{\beta}^{(2)}\hat{\beta}^{(1)}. \tag{6}$$

From Eqs. (4) and (6), we find

$$\tilde{A} = \hat{\beta}^{(3)}\hat{\beta}^{(2)}\hat{\beta}^{(1)}A\hat{\beta}^{(1)\prime}\hat{\beta}^{(2)\prime}\hat{\beta}^{(3)\prime}. \tag{7}$$

It follows from Eq. (7) that the general congruent transformation includes the orthogonal transformation $\hat{\beta}^{(1)}$, axial stretching with the matrix

$$\hat{\beta}^{(2)} = \operatorname{diag}(\beta_1^2,\ \beta_2^2,\ \beta_3^2,\ \beta_2\beta_3,\ \beta_1\beta_3,\ \beta_1\beta_2)$$

and one more orthogonal transformation $\hat{\beta}^{(3)}$.

Let us write Eqs. (1) for the two-dimensional case:

$$(A_{i(11)j}\,\partial_{11} + 2A_{i(12)j}\,\partial_{12} + A_{i(22)j}\,\partial_{22})u_j + F_i = 0, \qquad i, j = 1, 2. \tag{8}$$

With allowance for the notation for the elasticity moduli A_{ij} with two subscripts, the matrix of the operators in Eqs. (8) has the form

$$A_{i(kl)j}\,\partial_{kl} = \begin{bmatrix} A_{11}\,\partial_{11} + \sqrt{2}\,A_{61}\,\partial_{12} + \dfrac{A_{66}}{2}\,\partial_{22} & \dfrac{A_{61}}{\sqrt{2}}\,\partial_{11} + \Big(A_{21} + \dfrac{A_{66}}{2}\Big)\,\partial_{12} + \dfrac{A_{62}}{\sqrt{2}}\,\partial_{22} \\ \dfrac{A_{61}}{\sqrt{2}}\,\partial_{11} + \Big(A_{21} + \dfrac{A_{66}}{2}\Big)\,\partial_{12} + \dfrac{A_{62}}{\sqrt{2}}\,\partial_{22} & \dfrac{A_{66}}{2}\,\partial_{11} + \sqrt{2}\,A_{62}\,\partial_{12} + A_{22}\,\partial_{22} \end{bmatrix}.$$

Equations (8) are also presented in the form

$$\left(\begin{bmatrix} A_{11} & A_{61}/\sqrt{2} \\ A_{61}/\sqrt{2} & A_{66}/2 \end{bmatrix}\partial_{11} + \begin{bmatrix} \sqrt{2}\,A_{61} & A_{21}+A_{66}/2 \\ A_{21}+A_{66}/2 & \sqrt{2}\,A_{62} \end{bmatrix}\partial_{12} + \begin{bmatrix} A_{66}/2 & A_{62}/\sqrt{2} \\ A_{62}/\sqrt{2} & A_{22} \end{bmatrix}\partial_{22}\right) u_j + F_i = 0. \quad (9)$$

By virtue of the positive definiteness of the matrix of the elasticity moduli A_{ij}, the matrices $A_{i(11)j}$ and $A_{i(22)j}$ in Eqs. (8) and (9) are also positively defined, as well as the matrix $A_{i(kl)j}\partial_{kl}$ for all non-zero real values of the symbols ∂_k and $\partial_{11} + \partial_{22} \neq 0$. Under the affine transformations (2), the form of Eqs. (9) is unchanged. All moduli A_{ij} can be assumed to be dimensionless. This means that the stresses and moduli are normalized to a certain fixed stress.

Using consecutive transformations of the form (5), (6), we can obtain the following values of the coefficients in Eqs. (9):

$$A_{61} = 0, \quad A_{62} = 0; \qquad A_{11} = 1, \quad A_{66} = 2$$

(the tilde is omitted). Equations (9) acquire the form

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\partial_{11} + \begin{bmatrix} 0 & 1+A_{21} \\ 1+A_{21} & 0 \end{bmatrix}\partial_{12} + \begin{bmatrix} 1 & 0 \\ 0 & A_{22} \end{bmatrix}\partial_{22}\right) u_j + F_i = 0$$

or

$$\begin{bmatrix} \partial_{11}+\partial_{22} & (1+A_{21})\,\partial_{12} \\ (1+A_{21})\,\partial_{12} & \partial_{11}+A_{22}\,\partial_{22} \end{bmatrix} u_j + F_i = 0. \quad (10)$$

It follows from the considerations above that an arbitrary matrix of the elasticity moduli

$$A_{ij} = \begin{bmatrix} A_{11} & & \text{sym} \\ A_{21} & A_{22} & \\ A_{61} & A_{62} & A_{66} \end{bmatrix}$$

with the use of the congruent transformations (7) caused by the affine transformations (2) and (5) can be always brought to the canonical form

$$A_{ij} = \begin{bmatrix} 1 & & \text{sym} \\ A_{21} & A_{22} & \\ 0 & 0 & 2 \end{bmatrix}, \quad (11)$$

and Eqs. (8) and (9) can be always brought to the form (10) [9]. The elasticity moduli A_{21} and A_{22} in Eqs. (10), (11) will be called the canonical moduli. Thus, the two-dimensional problem of the linear elasticity theory for an arbitrary anisotropic material reduces to solving Eqs. (10) with the boundary conditions in stresses or displacements.

Some other canonical forms of equations and elasticity moduli for a two-dimensional problem were considered in [10–13].

Let us demonstrate that Eqs. (8) and (9) are converted to the form (10) by means of transformations of the form (2), (5)–(7), and the moduli A_{ij} are converted to the form (11). The determinant of the matrix of the operators in Eqs. (8) and (9) is

$$d = |A_{i(kl)j}\,\partial_{kl}| = (1/2)(A_{11}A_{66} - A_{61}^2)\,\partial_{1111} + \sqrt{2}\,(A_{11}A_{62} - A_{21}A_{61})\,\partial_{1112}$$
$$+ (A_{11}A_{22} - A_{21}^2 - A_{21}A_{66} + A_{61}A_{62})\,\partial_{1122} + \sqrt{2}\,(A_{22}A_{61} - A_{21}A_{62})\,\partial_{1222}$$
$$+ (1/2)(A_{22}A_{66} - A_{62}^2)\,\partial_{2222} > 0 \quad (12)$$

for all non-zero real values of the symbols ∂_k and $\partial_{11} + \partial_{22} \neq 0$. Determinant (12) has the same form in the transformed coordinate system. As we always have $d > 0$, then system (8) or (9) is elliptical, and the equation $|A_{i(kl)j}\, \partial_{kl}| = 0$ has no real roots; determinant (12) is decomposed in this case into the quadratic multipliers

$$d = (a_{11}\,\partial_{11} + 2a_{12}\,\partial_{12} + a_{22}\,\partial_{22})(b_{11}\,\partial_{11} + 2b_{12}\,\partial_{12} + b_{22}\,\partial_{22}) = D_1 D_2, \tag{13}$$

which have no real roots either. Comparing the coefficients in Eqs. (12) and (13), we obtain the equations

$$\begin{gathered}
(A_{11}A_{66} - A_{61}^2)/2 = a_{11}b_{11}, \\
\sqrt{2}\,(A_{11}A_{62} - A_{21}A_{61}) = 2(a_{12}b_{11} + a_{11}b_{12}), \\
A_{11}A_{22} - A_{21}^2 - A_{21}A_{66} + A_{61}A_{62} = a_{22}b_{11} + 4a_{12}b_{12} + a_{11}b_{22}, \\
\sqrt{2}\,(A_{22}A_{61} - A_{21}A_{62}) = 2(a_{22}b_{12} + a_{12}b_{22}), \\
(A_{22}A_{66} - A_{62}^2)/2 = a_{22}b_{22}.
\end{gathered} \tag{14}$$

If A_{ij} are given, we can determine $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ from Eqs. (14); the coefficient a_{11} can be fixed arbitrarily.

Let transformation (5) be such that the coefficients at the symbols ∂_{1112} and ∂_{1222} in Eqs. (12) are equal to zero (the tilde is omitted):

$$A_{11}A_{62} - A_{21}A_{61} = 0, \qquad A_{22}A_{61} - A_{21}A_{62} = 0. \tag{15}$$

By virtue of positive definiteness, the determinant of system (15) is $A_{11}A_{22} - A_{21}^2 > 0$; therefore, it follows from Eqs. (15) that $A_{61} = 0$ and $A_{62} = 0$. With allowance for Eqs. (3), the last equalities are written in the form

$$\tilde{A}_{2111} = \tilde{A}_{61}/\sqrt{2} = \beta_{2i}\beta_{1j}A_{ijkl}\beta_{1k}\beta_{1l} = 0, \quad \tilde{A}_{2122} = \tilde{A}_{62}/\sqrt{2} = \beta_{2i}\beta_{1j}A_{ijkl}\beta_{2k}\beta_{2l} = 0. \tag{16}$$

The existence of the solution β_{pi} of Eqs. (16) is proved with a rather complicated procedure in [10–13]. Assuming that

$$\beta_{pi} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{bmatrix} = \begin{bmatrix} \beta_1 n_{11} & \beta_1 n_{21} \\ \beta_2 n_{12} & \beta_2 n_{22} \end{bmatrix},$$

where $n_{ik}n_{il} = \delta_{kl}$, we write Eqs. (16) in the form

$$n_{i2}A_{ijkl}n_{j1}n_{k1}n_{l1} = 0, \qquad n_{i1}A_{ijkl}n_{j2}n_{k2}n_{l2} = 0. \tag{17}$$

Equations (17) are conditions of existence of longitudinal normals, which occur for an arbitrary tensor A_{ijkl} of the elasticity moduli [14].

The solution of Eqs. (17) is equivalent to determining the directions n_{i1} and n_{i2}, for which the quantities

$$\begin{aligned}
\tilde{A}_{1111} = \tilde{A}_{11} = A_{ijkl}n_{i1}n_{j1}n_{k1}n_{l1}, \qquad n_{i1}n_{i1} = 1, \\
\tilde{A}_{2222} = \tilde{A}_{22} = A_{ijkl}n_{i2}n_{j2}n_{k2}n_{l2}, \qquad n_{i2}n_{i2} = 1
\end{aligned} \tag{18}$$

reach extreme values. The conditions of existence of extreme points of the quantities (18) [14]

$$A_{ijkl}n_{j1}n_{k1}n_{l1} = \tilde{A}_{11}n_{i1}, \qquad A_{ijkl}n_{j2}n_{k2}n_{l2} = \tilde{A}_{22}n_{i2}$$

yield Eqs. (17). Relations (18) are written in a more detailed form as

$$\tilde{A} = A_{11}n_1^4 + 2\sqrt{2}\,A_{61}n_1^3n_2 + 2(A_{21} + A_{66})n_1^2n_2^2 + 2\sqrt{2}\,A_{62}n_1n_2^3 + A_{22}n_2^4, \quad n_1^2 + n_2^2 = 1. \tag{19}$$

Assuming that $n_1 = c$ and $n_2 = s$ or $n_1 = -s$, $n_2 = c$, and $c^2 + s^2 = 1$ in Eqs. (19), we obtain Eqs. (17) in a detailed form as the extremum condition for the quantity $\tilde{A}$:

$$\begin{gathered}
A_{61}c^4 + \sqrt{2}\,(A_{21} + A_{66} - A_{11})c^3s + 3(A_{62} - A_{61})c^2s^2 \\
+ \sqrt{2}\,(A_{22} - A_{21} - A_{66})cs^3 - A_{62}s^4 = 0, \\
A_{62}c^4 + \sqrt{2}\,(A_{22} - A_{21} - A_{66})c^3s + 3(A_{61} - A_{62})c^2s^2 + \sqrt{2}\,(A_{21} + A_{66} - A_{11})cs^3 - A_{61}s^4 = 0.
\end{gathered} \tag{20}$$

As the second equation in system (20) is obtained from the first one with the use of the substitutions $c \to -s$ and $s \to c$, it is sufficient to solve anyone of these equations. For the variable $t = s/c$, the first equation in system (20) yields the fourth-order equation

$$A_{61} + \sqrt{2}\,(A_{21} + A_{66} - A_{11})t + 3(A_{62} - A_{61})t^2 + \sqrt{2}\,(A_{22} - A_{21} - A_{66})t^3 - A_{62}t^4 = 0,$$

which has real roots for all A_{ij}, because there are always extreme values of form (19) [14]. All quantities $A_{ij}^{(1)}$ are expressed by Eq. (7) via A_{ij} and the root t.

Thus, it follows from the above-given considerations that there always exists an *orthogonal* transformation $\beta_{pi}^{(1)} = n_{ip}$ converting an arbitrary matrix of the elasticity moduli A_{ij} to the form

$$A_{ij}^{(1)} = \begin{bmatrix} A_{11} & & \text{sym} \\ A_{21} & A_{22} & \\ 0 & 0 & A_{66} \end{bmatrix}. \tag{21}$$

After that, with the use of the second transformation $\beta^{(2)} = \operatorname{diag}(\beta_1,\ \beta_2)$, matrix (21) is brought to the form

$$A_{ij}^{(2)} = \begin{bmatrix} \beta_1^4 A_{11} & & \text{sym} \\ \beta_1^2\beta_2^2 A_{21} & \beta_2^4 A_{22} & \\ 0 & 0 & \beta_1^2\beta_2^2 A_{66} \end{bmatrix}.$$

Choosing the parameters β_1 and β_2, we can obtain from the matrix $A_{ij}^{(2)}$ either the canonical matrix (11), where $A_{11}^{(2)} = \beta_1^4 A_{11} = 1$ and $A_{66}^{(2)} = \beta_1^2\beta_2^2 A_{66} = 2$, the matrix [10]

$$A_{ij}^{(2)} = \begin{bmatrix} 1 & & \text{sym} \\ \beta - \alpha & 1 & \\ 0 & 0 & 2\alpha \end{bmatrix}, \tag{22}$$

where

$$\beta_1^2 = \frac{1}{\sqrt{A_{11}}}, \quad \beta_2^2 = \frac{1}{\sqrt{A_{22}}}, \quad \beta - \alpha = \frac{A_{21}}{\sqrt{A_{11}A_{22}}}, \quad 2\alpha = \frac{A_{66}}{\sqrt{A_{11}A_{22}}},$$

or the matrix [11, 12], where $A_{11}^{(2)} = A_{22}^{(2)} = \beta_1^4 A_{11} = \beta_2^4 A_{22}$ and $\beta_1\beta_2 = 1$.

With allowance for Eqs. (21), Eqs. (14) acquire the form

$$\begin{gathered} a_{11}b_{11} = \frac{1}{2}\,A_{11}A_{66}, \qquad \frac{a_{12}}{a_{11}} + \frac{b_{12}}{b_{11}} = 0, \qquad \frac{a_{22}}{a_{11}}\,\frac{b_{12}}{b_{11}} + \frac{a_{12}}{a_{11}}\,\frac{b_{22}}{b_{11}} = 0, \\ \frac{a_{22}}{a_{11}} + 4\,\frac{a_{12}}{a_{11}}\,\frac{b_{12}}{b_{11}} + \frac{b_{22}}{b_{11}} = \frac{A_{11}A_{22} - A_{21}^2 - A_{21}A_{66}}{A_{11}A_{66}/2}, \qquad \frac{a_{22}}{a_{11}}\,\frac{b_{22}}{b_{11}} = \frac{A_{22}}{A_{11}}. \end{gathered} \tag{23}$$

From the second and third equations of system (23), we find

$$\frac{b_{12}}{b_{11}} = -\frac{a_{12}}{a_{11}}, \qquad \frac{a_{12}}{a_{11}}\Big(\frac{b_{22}}{b_{11}} - \frac{a_{22}}{a_{11}}\Big) = 0,$$

whence it follows that two variants are possible:

$$\frac{a_{12}}{a_{11}} = 0; \tag{24a}$$

$$\frac{b_{22}}{b_{11}} = \frac{a_{22}}{a_{11}}. \tag{24b}$$

If variant (24a) is realized, then it follows from the fourth and fifth equations of system (23) that a_{22}/a_{11} and b_{22}/b_{11} are roots of the quadratic equation

$$a^2 - \frac{A_{11}A_{22} - A_{21}^2 - A_{21}A_{66}}{A_{11}A_{66}/2}\,a + \frac{A_{22}}{A_{11}} = 0.$$

From here, we find

$$a_{1,2} = \frac{A_{11}A_{22} - A_{21}^2 - A_{21}A_{66}}{A_{11}A_{66}} \pm \sqrt{\frac{(A_{11}A_{22} - A_{21}^2 - A_{21}A_{66})^2}{(A_{11}A_{66})^2} - \frac{A_{22}}{A_{11}}}$$

$$= \frac{1}{A_{11}A_{66}}\Big[A_{11}A_{22} + \Big(\frac{1}{2}\,A_{66}\Big)^2 - \Big(A_{21} + \frac{1}{2}\,A_{66}\Big)^2$$

$$\pm\sqrt{(A_{11}A_{22} - A_{21}^2)[A_{11}A_{22} - (A_{21} + A_{66})^2]}\,\Big]. \tag{25}$$

Roots (25) are real and positive if the condition of positive definiteness of matrix (21)

$$-\sqrt{A_{22}/A_{11}} < A_{21}/A_{11} < \sqrt{A_{22}/A_{11}} \tag{26}$$

and the inequality

$$A_{21}/A_{11} \leq \sqrt{A_{22}/A_{11}} - A_{66}/A_{11} \tag{27}$$

are satisfied. If the inequality

$$\sqrt{A_{22}/A_{11}} - A_{66}/A_{11} < A_{21}/A_{11} \tag{28}$$

is valid, we have variant (24b). Then, the two last equations in system (23) yield the coefficients

$$a_{22}/a_{11} = \sqrt{A_{22}/A_{11}},$$

$$2a_{12}/a_{11} = \sqrt{2\Big[\sqrt{A_{22}/A_{11}} - (A_{11}A_{22} - A_{21}^2 - A_{21}A_{66})/(A_{11}A_{66})\Big]} \tag{29}$$

$$= \sqrt{2\Big(\sqrt{A_{11}A_{22}} + A_{21}\Big)\Big[A_{66} - \Big(\sqrt{A_{11}A_{22}} - A_{21}\Big)\Big]/(A_{11}A_{66})}.$$

Thus, if inequalities (26)–(28) are satisfied, the determinant of Eq. (12)

$$d = \frac{1}{2}\,A_{11}A_{66}\,\partial_{1111} + (A_{11}A_{22} - A_{21}^2 - A_{21}A_{66})\,\partial_{1122} + \frac{1}{2}\,A_{22}A_{66}\,\partial_{2222} \tag{30}$$

is decomposed into factors of the form (13)

$$d = a_{11}b_{11}(\partial_{11} + a_1\,\partial_{22})(\partial_{11} + a_2\,\partial_{22}); \tag{31a}$$

$$d = a_{11}b_{11}\Big(\partial_{11} + 2\,\frac{a_{12}}{a_{11}}\,\partial_{12} + \sqrt{\frac{A_{22}}{A_{11}}}\,\partial_{22}\Big)\Big(\partial_{11} - 2\,\frac{a_{12}}{a_{11}}\,\partial_{12} + \sqrt{\frac{A_{22}}{A_{11}}}\,\partial_{22}\Big), \tag{31b}$$

where the coefficients are determined above [see Eqs. (25) and (29)]. The quadratic forms in Eq. (31b) are positively determined because the inequality $(a_{12}/a_{11})^2 < \sqrt{A_{22}/A_{11}}$ is satisfied. If we have the equality sign in Eq. (27), then Eq. (31a) yields

$$d = a_{11}b_{11}(\partial_{11} + \sqrt{A_{22}/A_{11}}\,\partial_{22})^2. \tag{32}$$

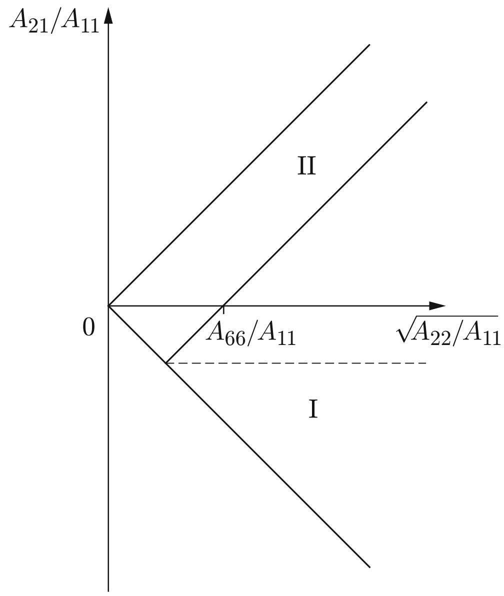

Fig. 1. Domains of admissible values of A_{ij}: I is the domain determined by inequalities (26) and (27); II is the domain determined by inequalities (26) and (28).

Thus, the presentation of determinant (30) in the form (31a), (31b), or (32) depends on which relations (26)–(28) are satisfied, i.e., in which domain of admissible values the elasticity moduli A_{ij} are located. The domain of admissible values of the moduli A_{ij} described by inequalities (26)–(28) is shown in Fig. 1. Domain I is obtained if inequalities (26) and (27) are satisfied, and domain II is obtained if inequalities (26) and (28) are satisfied. The equality sign in Eq. (27) corresponds to the straight line separating domains I and II in Fig. 1. In what follows, we assume that $A_{11} = A_{66}/2 = 1$ and $a_{11} = b_{11} = 1$.

Let us write Eqs. (9) for the case where the matrix A_{ij} has the form (22):

$$\begin{aligned} &(\partial_{11} + \alpha\,\partial_{22})u_1 + \beta\,\partial_{12}u_2 + F_1 = 0, \\ &\beta\,\partial_{12}u_1 + (\alpha\,\partial_{11} + \partial_{22})u_2 + F_2 = 0. \end{aligned} \tag{33}$$

The determinant of the matrix of the operators in system (33) is

$$\begin{aligned} d &= \alpha\,\partial_{1111} + (1 + \alpha^2 - \beta^2)\,\partial_{1122} + \alpha\,\partial_{2222} \\ &= (1/4)[(1+\alpha)^2 - \beta^2](\partial_{11} + \partial_{22})^2 + (1/4)[\beta^2 - (1-\alpha)^2](\partial_{11} - \partial_{22})^2. \end{aligned} \tag{34}$$

The determinant is $d > 0$, i.e., system (33) is elliptical if

$$-(1+\alpha) < \beta < 1 + \alpha, \qquad \alpha > 0. \tag{35}$$

At

$$-(1-\alpha) \le \beta \le 1 - \alpha, \qquad 0 < \alpha \le 1; \tag{36a}$$

$$-(\alpha - 1) \le \beta \le \alpha - 1, \qquad 1 < \alpha, \tag{36b}$$

determinant (34) is written in the form

$$d = \alpha(\partial_{11} + a_1\,\partial_{22})(\partial_{11} + a_2\,\partial_{22}), \tag{37}$$

where

$$a_{1,2} = \frac{1}{2\alpha}\left\{1 + \alpha^2 - \beta^2 \pm \sqrt{[(1+\alpha)^2 - \beta^2][(1-\alpha)^2 - \beta^2]}\,\right\}.$$

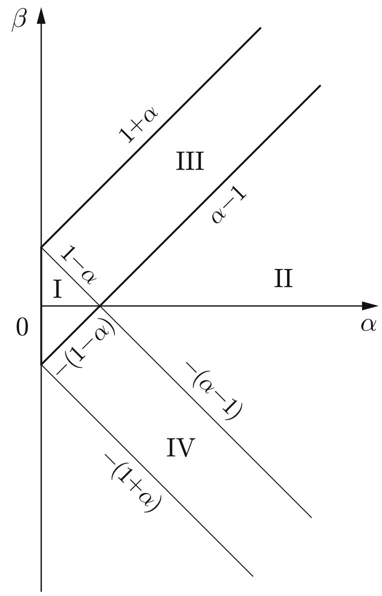

Fig. 2. Domains of admissible values of the parameters α and β: I is the domain determined by inequalities (36a); II is the domain determined by inequalities (36b); III is the domain determined by inequalities (38a); IV is the domain determined by inequalities (38b).

If the inequalities

$$1-\alpha<\beta, \quad 0<\alpha\le 1, \quad \alpha-1<\beta, \quad 1<\alpha; \tag{38a}$$

$$\beta<-(1-\alpha), \quad 0<\alpha\le 1, \quad \beta<-(\alpha-1), \quad 1<\alpha \tag{38b}$$

are satisfied, then determinant (34) is decomposed into the factors

$$d=\alpha\Big(\partial_{11}+2\,\frac{a_{12}}{a_{11}}\,\partial_{12}+\partial_{22}\Big)\Big(\partial_{11}-2\,\frac{a_{12}}{a_{11}}\,\partial_{12}+\partial_{22}\Big), \tag{39}$$

where

$$2a_{12}/a_{11}=\sqrt{[\beta^2-(1-\alpha)^2]/\alpha}.$$

Thus, depending on which of inequalities (36a), (36b), (38a), or (38b) is satisfied for the coefficients α and β, determinant (34) is presented in the form (37) or (39). The domains of admissible values of α and β are shown in Fig. 2. The domain of ellipticity of system (33) is determined by inequalities (35) and includes domains I, II, III, and IV in Fig. 2. The case $a_1=a_2$ in Eq. (37) corresponds to the equality signs in Eqs. (36a) and (36b) and to the straight lines $\beta=1-\alpha$ and $\beta=\alpha-1$ in Fig. 2.

In the discussion above, we ignored the condition of positive definiteness of the specific strain energy, i.e., matrix (22):

$$\alpha-1<\beta<1+\alpha, \qquad \alpha>0. \tag{40}$$

Domains I and III in Fig. 2 correspond to inequalities (40). Thus, if α and β belong to domains I and III in Fig. 2, then Eqs. (33) are the equations of the elasticity theory. If α and β are located in domains II and IV in Fig. 2, then Eqs. (33) are not the equations of the elasticity theory, because conditions (40) are not satisfied. Nevertheless, Eqs. (33) remain elliptical, and presentations (37) and (39) are valid. The domain of ellipticity of Eqs. (10) is also wider than the domain of elasticity (26).

For an isotropic material, matrix (11) and Eqs. (10) take the form

$$A_{ij}=\begin{bmatrix} 1 & & \text{sym} \\ \lambda/\mu & (\lambda/\mu+2)^2 & \\ 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} \partial_{11}+\partial_{22} & (\lambda/\mu+1)\,\partial_{12} \\ (\lambda/\mu+1)\,\partial_{12} & \partial_{11}+(\lambda/\mu+2)^2\partial_{22} \end{bmatrix} u_j+F_i=0$$

(λ and μ are the Lamé constants), and the determinant of the matrix of the operators is

$$d = [\partial_{11} + (\lambda/\mu + 2)\,\partial_{22}]^2.$$

The parameters α and β in Eqs. (33) are [10]

$$\alpha = \frac{\mu}{\lambda + 2\mu}, \qquad \beta = \frac{\lambda + \mu}{\lambda + 2\mu}, \qquad \alpha + \beta = 1.$$

Let $D = \mathrm{diag}\,(D_1,\ D_2)$ be a diagonal matrix where D_1 and D_2 are the multipliers in Eqs. (31) and (32), i.e., $d = D_1 D_2$. For variants (31a), (31b), and (32), there exist matrices C and B,

$$C = \begin{bmatrix} \alpha_{111}\,\partial_1 + \alpha_{121}\,\partial_2 & \alpha_{112}\,\partial_1 + \alpha_{122}\,\partial_2 \\ \alpha_{211}\,\partial_1 + \alpha_{221}\,\partial_2 & \alpha_{212}\,\partial_1 + \alpha_{222}\,\partial_2 \end{bmatrix},$$
$$B = \begin{bmatrix} \beta_{111}\,\partial_1 + \beta_{121}\,\partial_2 & \beta_{112}\,\partial_1 + \beta_{122}\,\partial_2 \\ \beta_{211}\,\partial_1 + \beta_{221}\,\partial_2 & \beta_{212}\,\partial_1 + \beta_{222}\,\partial_2 \end{bmatrix}, \tag{41}$$

such that the following relation holds:

$$AC = BD \tag{42}$$

[A is the matrix of the operators in (10)]. As $|A||C| = |B||D|$ and $|A| = |D| = d = D_1 D_2$, then we have $|C| = |B|$. It was shown [15] that the general solution of the homogeneous equations (10) $Au = 0$ is presented in the form

$$u = C\varphi, \qquad D\varphi = f, \qquad Bf = 0. \tag{43}$$

The formulas $u = C\varphi$, $\varphi = B'\tilde{u}$, and $A\tilde{u} = 0$ transform the solutions of the equations $Au = 0$ and $D\varphi = 0$ to each other. In this procedure, $u = CB'\tilde{u}$ is the formula for production of new solutions, i.e., $Q = CB'$ is the symmetry operator [15].

For Eq. (31a), matrices (41) have the form

$$C = \begin{bmatrix} \alpha_{111}\,\partial_1 + a_1\beta_{121}\,\partial_2 & \alpha_{112}\,\partial_1 + a_2\beta_{122}\,\partial_2 \\ \alpha_{211}\,\partial_1 + a_1\psi_1\,\partial_2 & \alpha_{212}\,\partial_1 + a_2\psi_2\,\partial_2 \end{bmatrix},$$
$$B = \begin{bmatrix} \alpha_{111}\,\partial_1 + \beta_{121}\,\partial_2 & \alpha_{112}\,\partial_1 + \beta_{122}\,\partial_2 \\ \alpha_{211}\,\partial_1 + A_{22}\psi_1\,\partial_2 & \alpha_{212}\,\partial_1 + A_{22}\psi_2\,\partial_2 \end{bmatrix}. \tag{44}$$

Here,

$$\beta_{121} = (A_{22} - a_1)\alpha_1 - (1 + A_{21})\alpha_2, \qquad \alpha_{211} = -(1 + A_{21})a_1\alpha_1 + (a_1 - 1)\alpha_2,$$

$$\alpha_{111} = (a_1 - A_{22})\beta_1 - (1 + A_{21})a_1\beta_2, \qquad \psi_1 = -(1 + A_{21})\beta_1 + (1 - a_1)\beta_2,$$

$$\beta_{122} = (A_{22} - a_1)\gamma_1 - (1 + A_{21})\gamma_2, \qquad \alpha_{212} = -(1 + A_{21})a_2\gamma_1 + (a_2 - 1)\gamma_2,$$

$$\alpha_{112} = (a_2 - A_{22})\delta_1 - (1 + A_{21})a_2\delta_2, \qquad \psi_2 = -(1 + A_{21})\delta_1 + (1 - a_2)\delta_2,$$

α_i, β_i, γ_i, and δ_i are free parameters. By means of direct verification, we can show that relation (42) is valid for matrices (44). If $1 + A_{21} = 0$ ($A_{21} + A_{66}/2 = 0$), then system (10) is already diagonal. The straight line $1 + A_{21} = 0$ corresponds to the dashed line in Fig. 1.

If Eq. (32) is valid, then matrices (41) take the form

$$C = \begin{bmatrix} \alpha_{111}\,\partial_1 - \sqrt{A_{22}}\,\alpha_{211}\,\partial_2 & \alpha_{112}\,\partial_1 - \sqrt{A_{22}}\,\alpha_{212}\,\partial_2 \\ \alpha_{211}\,\partial_1 + \alpha_{111}\,\partial_2 & \alpha_{212}\,\partial_1 + \alpha_{112}\,\partial_2 \end{bmatrix},$$
$$B = \begin{bmatrix} \alpha_{111}\,\partial_1 - \alpha_{211}\,\partial_2 & \alpha_{112}\,\partial_1 - \alpha_{212}\,\partial_2 \\ \alpha_{211}\,\partial_1 + \sqrt{A_{22}}\,\alpha_{111}\,\partial_2 & \alpha_{212}\,\partial_1 + \sqrt{A_{22}}\,\alpha_{112}\,\partial_2 \end{bmatrix}, \tag{45}$$

and the parameters α_{111}, α_{211}, α_{112}, and α_{212} remain free. The determinants of matrices (45) coincide:

$$|C| = |B| = (\alpha_{111}\alpha_{212} - \alpha_{211}\alpha_{112})(\partial_{11} + \sqrt{A_{22}}\,\partial_{22})$$

and do not vanish if $\alpha_{111}\alpha_{212} - \alpha_{211}\alpha_{112} \neq 0$. Relation (42) is also satisfied.

For Eq. (31b), matrices (41) have the form

$$C = \begin{bmatrix} \alpha_{111}\,\partial_1 + \sqrt{A_{22}}\,\beta_{121}\,\partial_2 & \alpha_{112}\,\partial_1 + \sqrt{A_{22}}\,\beta_{122}\,\partial_2 \\ \alpha_{211}\,\partial_1 + \alpha_{221}\,\partial_2 & \alpha_{212}\,\partial_1 + \alpha_{222}\,\partial_2 \end{bmatrix},$$

$$B = \begin{bmatrix} \alpha_{111}\,\partial_1 + \beta_{121}\,\partial_2 & \alpha_{112}\,\partial_1 + \beta_{122}\,\partial_2 \\ \alpha_{211}\,\partial_1 + \sqrt{A_{22}}\,\alpha_{221}\,\partial_2 & \alpha_{212}\,\partial_1 + \sqrt{A_{22}}\,\alpha_{222}\,\partial_2 \end{bmatrix}. \tag{46}$$

Here, we have

$$\alpha_{211} = \frac{1}{1 + A_{21}}\Big[\Big(1 - \sqrt{A_{22}}\,\Big)\beta_{121} + 2a_{12}\alpha_{111}\Big],$$

$$\alpha_{221} = \frac{1}{1 + A_{21}}\Big[2a_{12}\beta_{121} + \Big(\sqrt{A_{22}} - 1\Big)\alpha_{111}\Big],$$

$$\alpha_{212} = \frac{1}{1 + A_{21}}\Big[\Big(1 - \sqrt{A_{22}}\,\Big)\beta_{122} - 2a_{12}\alpha_{112}\Big],$$

$$\alpha_{222} = \frac{1}{1 + A_{21}}\Big[-2a_{12}\beta_{122} + \Big(\sqrt{A_{22}} - 1\Big)\alpha_{112}\Big];$$

the coefficients α_{111}, β_{121}, α_{112}, and β_{122} remain arbitrary, and relation (42) is also satisfied.

In constructing matrices (41), some coefficients in Eqs. (44)–(46) remain free, which allows obtaining various forms of presentation (43) of displacements via the quasi-harmonic functions φ_1 and φ_2. Let us give some examples.

For variant (31a), Eqs. (44) can be used to obtain the matrices C and B in the form

$$C = \begin{bmatrix} \partial_1 & \partial_1 \\ k_1\,\partial_2 & k_2\,\partial_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \partial_1 & \partial_1 \\ a_2k_1\,\partial_2 & a_1k_2\,\partial_2 \end{bmatrix},$$

$$|C| = |B| = \frac{a_2 - a_1}{1 + A_{21}}\,\partial_{12}, \qquad k_1 = \frac{a_1 - 1}{1 + A_{21}}, \qquad k_2 = \frac{a_2 - 1}{1 + A_{21}},$$

or

$$C = \begin{bmatrix} -a_1k_2\,\partial_2 & -a_2k_1\,\partial_2 \\ \partial_1 & \partial_1 \end{bmatrix}, \quad B = \begin{bmatrix} -k_2\,\partial_2 & -k_1\,\partial_2 \\ \partial_1 & \partial_1 \end{bmatrix}, \quad |C| = |B| = \frac{a_1 - a_2}{1 + A_{21}}\,\partial_{12},$$

or

$$C = \begin{bmatrix} \partial_1 & -a_2k_1\,\partial_2 \\ k_1\,\partial_2 & \partial_1 \end{bmatrix}, \quad B = \begin{bmatrix} \partial_1 & -k_1\,\partial_2 \\ a_2k_1\,\partial_2 & \partial_1 \end{bmatrix}, \quad |C| = |B| = \partial_{11} + a_2k_1^2\,\partial_{22},$$

or

$$C = \begin{bmatrix} -a_1k_2\,\partial_2 & -\partial_1 \\ \partial_1 & -k_2\,\partial_2 \end{bmatrix}, \quad B = \begin{bmatrix} -k_2\,\partial_2 & -\partial_1 \\ \partial_1 & -a_1k_2\,\partial_2 \end{bmatrix}, \quad |C| = |B| = \partial_{11} + a_1k_2^2\,\partial_{22}.$$

The functions φ_1 and φ_2 satisfy the equations

$$(\partial_{11} + a_1\,\partial_{22})\varphi_1 = f_1, \qquad (\partial_{11} + a_2\,\partial_{22})\varphi_2 = f_2, \qquad Bf = 0, \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

For variant (32), Eqs. (45) can be used to obtain the matrices C and B in the form

$$C = \begin{bmatrix} \partial_1 & -\sqrt{A_{22}}\,\partial_2 \\ \partial_2 & \partial_1 \end{bmatrix}, \qquad B = \begin{bmatrix} \partial_1 & -\partial_2 \\ \sqrt{A_{22}}\,\partial_2 & \partial_1 \end{bmatrix},$$

or

$$C = \begin{bmatrix} \sqrt{A_{22}}\,\partial_2 & \partial_1 \\ -\partial_1 & \partial_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \partial_2 & \partial_1 \\ -\partial_1 & \sqrt{A_{22}}\,\partial_2 \end{bmatrix}, \qquad |C| = |B| = \partial_{11} + \sqrt{A_{22}}\,\partial_{22}.$$

The functions φ_1 and φ_2 satisfy the equations

$$(\partial_{11} + \sqrt{A_{22}}\,\partial_{22})\varphi_1 = f_1, \qquad (\partial_{11} + \sqrt{A_{22}}\,\partial_{22})\varphi_2 = f_2, \qquad Bf = 0.$$

For variant (31b), Eqs. (46) can be used to obtain the matrices C and B in the form

$$C = \begin{bmatrix} \partial_1 + m_2\sqrt{A_{22}}\,\partial_2 & \partial_1 - m_2\sqrt{A_{22}}\,\partial_2 \\ -m_1\,\partial_1 & m_1\,\partial_1 \end{bmatrix}, \qquad B = \begin{bmatrix} \partial_1 + m_2\,\partial_2 & \partial_1 - m_2\,\partial_2 \\ -m_1\,\partial_1 & m_1\,\partial_1 \end{bmatrix},$$

$$|C| = |B| = 2m_1\,\partial_{11}, \qquad m_1 = -\frac{1 + A_{21}}{2a_{12}}, \qquad m_2 = \frac{1 - \sqrt{A_{22}}}{2a_{12}}.$$

The functions φ_1 and φ_2 satisfy the equations

$$(\partial_{11} + 2a_{12}\,\partial_{12} + \sqrt{A_{22}}\,\partial_{22})\varphi_1 = f_1, \quad (\partial_{11} - 2a_{12}\,\partial_{12} + \sqrt{A_{22}}\,\partial_{22})\varphi_2 = f_2, \quad Bf = 0.$$

The above-given examples do not exhaust all possibilities of choosing matrices (44)–(46) for presenting the displacements $u = C\varphi$ via the quasi-harmonic functions φ_1 and φ_2 satisfying the equations $D_1\varphi_1 = f_1$, $D_2\varphi_2 = f_2$, and $Bf = 0$. In solving particular problems, the boundary conditions in displacements or stresses can also be simplified owing to the presence of free parameters in matrices (44)–(46).

Thus, the above-considered approach allows the two-dimensional problem of the linear elasticity theory for an arbitrary anisotropic material to be reduced to solving Eqs. (10) or (33) with two canonical moduli. With the use of relations (31a), (31b), (32) or (37), (39), and (42), systems (10) or (33) are converted to a diagonal form, and their general solution is presented in the form (43) via the quasi-harmonic functions φ_1 and φ_2 (see the examples given above). An isotropic material corresponds to relations (32), (45), which directly yield a presentation of displacements via the Kolosov–Muskhelishvili complex potentials. Other approaches to constructing the solution of Eqs. (9) were considered in [16].

In a three-dimensional case, it is also possible to simplify Eqs. (1) with the use of transformations (2), (4), and the matrix of the elasticity moduli A_{ij} with 21 independent components can be brought to a matrix with 12 independent canonical moduli. Generalizing Eqs. (17) to a three-dimensional case, we can show that there exists an orthogonal coordinate system (orthogonal transformation n_{ik}, $n_{ik}n_{il} = \delta_{kl}$) where $A_{42} = 0$, $A_{43} = 0$, $A_{51} = 0$, $A_{53} = 0$, $A_{61} = 0$, and $A_{62} = 0$. This means that there are no more than 15 independent elasticity moduli in the case of arbitrary anisotropy, rather than 21 moduli, as was believed previously. One variant of the canonical matrix with 12 moduli is given below:

$$A_{ij} = \begin{bmatrix} 1 & & & & & \\ A_{21} & 1 & & & \text{sym} & \\ A_{31} & A_{32} & 1 & & & \\ A_{41} & 0 & 0 & A_{44} & & \\ 0 & A_{52} & 0 & A_{54} & A_{55} & \\ 0 & 0 & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix}.$$

It follows from the form of this matrix that the assumption [6] about the congruency (equivalence) of an arbitrary matrix of the elasticity moduli A_{ij} to the matrix of the moduli of a material with monoclinic symmetry is not valid. The matrix A_{ij} of the form indicated above can be found in [17, 18]. The solution of Eqs. (1) in a three-dimensional case is also presented via three independent quasi-harmonic functions, i.e., system (1) is brought to a diagonal form.

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REFERENCES

1. N. I. Ostrosablin, "Affine transformations of the equations of the linear theory of elasticity," *J. Appl. Mech. Tech. Phys.*, **47**, No. 4, 564–572 (2006).
2. V. V. Kolokol'chikov, "Anisotropy reduced in solving problems to fictitious isotropy by tensor transformations," *Dokl. Akad. Nauk SSSR*, **300**, No. 3, 567–570 (1988).
3. G. Menditto, L. Quattrini, and A. M. Tarantino, "The contact problem for a class orthotropic elastic solids," *J. Elast.*, **33**, No. 2, 167–190 (1993).
4. G. W. Milton and A. B. Movchan, "A correspondence between plane elasticity and the two-dimensional real and complex dielectric equations in anisotropic media," *Proc. Roy. Soc. London, Ser. A*, **450**, No. 1939, 293–317 (1995).
5. A. Pouya and A. Zaoui, "A transformation of elastic boundary value problems with application to anisotropic behavior," *Int. J. Solids Struct.*, **43**, No. 16, 4937–4956 (2006).
6. S. Langer, S. A. Nazarov, and M. Specovius-Neugebauer, "Affine transforms of three-dimensional anisotropic media and explicit formulas for fundamental matrices," *J. Appl. Mech. Tech. Phys.*, **47**, No. 2, 229–235 (2006).
7. A. Pouya, "Green's function solution and displacement potentials for transformed transversely isotropic materials," *Europ. J. Mech., A: Solids*, **26**, No. 3, 491–502 (2007).
8. S. K. Godunov, *Elements of Continuum Mechanics* [in Russian], Nauka, Moscow (1978).
9. N. I. Ostrosablin, "Canonical form of equations of a plane static problem of anisotropic elasticity," in: *Problems of Mechanics of Continuous Media and Physics of Explosion*, Abstracts of All-Russia Conf. Devoted to the 50th Anniversary of the Lavrent'ev Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci. (Novosibirsk, September 17–22, 2007), Inst. Hydrodynamics, Novosibirsk (2007), p. 136.
10. P. J. Olver, "Canonical elastic moduli," *J. Elast.*, **19**, No. 3, 189–212 (1988).
11. N. B. Alfutova, A. B. Movchan, and S. A. Nazarov, "Algebraic equivalence of plane problems for orthotropic and anisotropic media," *Vestn. Leningr. Gos. Univ., Ser. 1*, **3**, No. 15, 64–68 (1991).
12. A. A. Kulikov, S. A. Nazarov, and M. A. Narbut, "Affine transforms in a plane problem of the elasticity theory," *Vestn. Leningr. Gos. Univ., Ser. 1*, **2**, No. 8, 91–95 (2000).
13. X. Li, B. Xu, and M. Wang, "On the canonical elastic moduli of linear plane anisotropic elasticity," *J. Elast.*, **85**, No. 1, 107–117 (2006).
14. F. I. Fedorov, *Theory of Elastic Waves in Crystals* [in Russian], Nauka, Moscow (1965).
15. N. I. Ostrosablin, "Symmetry operators and general solutions of the equations of the linear theory of elasticity," *J. Appl. Mech. Tech. Phys.*, **36**, No. 5, 724–729 (1995).
16. X.-L. Gao and R. E. Rowlands, "On displacement methods in planar anisotropic elasticity," *Mech. Res. Comm.*, **27**, No. 5, 553–560 (2000).
17. N. I. Ostrosablin, "Eigenoperators and eigenvectors for a system of differential equations of the linear theory of elasticity of anisotropic materials," *Dokl. Ross. Akad. Nauk*, **337**, No. 5, 608–610 (1994).
18. N. I. Ostrosablin, "Elastic anisotropic material with purely longitudinal and transverse waves," *J. Appl. Mech. Tech. Phys.*, **44**, No. 2, 271–278 (2003).